Mathematical Methods:
Summary of useful mathematical tools

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1 Introduction

This sheet is a summary of foundation material for the Mathematical Methods course (145). You will also find it useful for courses in the 2nd, 3rd and 4th year. Note that this is **not a formula sheet that you will be allowed in the exam**. You should ensure that you understand and recall everything on this sheet over the course of the term. If you have any questions about any of the formulae here, your Methods tutor will be happy to help.

2 Algebra

2.1 Quadratic formula

Solving $ax^2 + bx + c = 0$.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} &= -\frac{c}{a} \qquad \text{:completing the square} \\ x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

The discriminant of the quadratic is $b^2 - 4ac$. If $b^2 - 4ac < 0$ then the roots of the quadratic are complex.

2.2 Binomial expansion

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots \\ &\quad \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r\end{aligned}$$

where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

2.3 Factor Theorem

For a polynomial $f(x)$, if $f(a) = 0$ for some a, then $(x - a)$ divides $f(x)$.

2.4 Sequences and Series

An *arithmetic sequence* is defined as:

$$\{a, a+d, a+2d, a+3d, \ldots\}$$

for some initial integer a and term difference d. So the nth term of an arithmetic sequence is:

$$a_n = a + (n-1)d \qquad \text{for } n \geq 1$$

An *arithmetic series* is the sum of an arithmetic sequence up to the nth term. So the sum of the first n terms of an arithmetic sequence is SA_n where:

$$SA_n = \sum_{i=1}^{n} a_n = \frac{n}{2}(2a + (n-1)d)$$

A *geometric sequence* is defined as:

$$\{a, ar, ar^2, ar^3, \ldots\}$$

for some initial integer a and common ratio r. So the nth term of a geometric sequence is:

$$g_n = ar^{n-1} \qquad \text{for } n \geq 1$$

A *geometric series* is the sum of a geometric sequence up to the nth term. So the sum of the first n terms of a geometric sequence, SP_n, is:

$$SP_n = \sum_{i=1}^{n} g_n = \frac{a(1-r^n)}{1-r}$$

If the absolute value of r is less than one, that is $|r| < 1$, then this series converges as $n \to \infty$ and:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

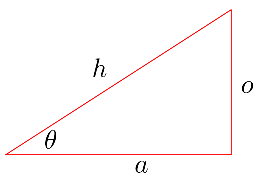
2.5 Partial Fractions

Especially useful for integration. The following tricks can be used to split up algebraic expressions with complex denominators.

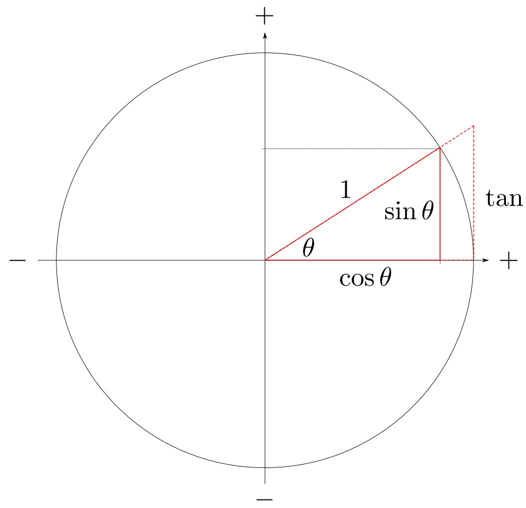
- $\dfrac{f(x)}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$
- $\dfrac{f(x)}{(x-a)(x-b)^2} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{(x-b)^2}$
- $\dfrac{f(x)}{(x-a)(x^2+b)} = \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+b}$

3 Trigonometry

Standard definitions: $\sin\theta = \frac{o}{h}$, $\cos\theta = \frac{a}{h}$ and $\tan\theta = \frac{o}{a}$



Given a circle of radius 1 with centre $(0, 0)$. The circumference of the circle can be represented by the coordinates $(\cos\theta, \sin\theta)$ for $0 \le \theta < 2\pi$ in radians (or $0 \le \theta < 360$ in degrees). The following sketch diagram can be used to remind yourself how $\cos\theta$ and $\sin\theta$ take positive and negative values as θ varies.



This also demonstrates $\cos^2\theta+\sin^2\theta = 1$ using Pythagoras on the circumscribed right angled triangle.

Below is a set of easy-to-remember sin, cos, tan results:

- $\sin 0 = 0 \quad \cos 0 = 1 \quad \tan 0 = 0$
- $\sin 30 = \sin\frac{\pi}{6} = \frac{1}{2} \quad \cos 30 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
- $\sin 45 = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \cos 45 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \tan 45 = \tan\frac{\pi}{4} = 1$
- $\sin 60 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos 60 = \cos\frac{\pi}{3} = \frac{1}{2}$
- $\sin 90 = \sin\frac{\pi}{2} = 1 \quad \cos 90 = \cos\frac{\pi}{2} = 0 \quad \tan 90 = \tan\frac{\pi}{2} = \infty$
- $\sin n\pi = 0 \quad \cos n\pi = (-1)^n \quad \tan n\pi = 0$

In the following set of formulae, the equations marked with an $(*)$ can be derived from the ones marked with a solid bullet.

$*$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

- $\sec\theta = \frac{1}{\cos\theta} \quad \mathrm{cosec}\,\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta}$
- $\cos^2\theta + \sin^2\theta = 1$

$*$ $1 + \tan^2\theta = \sec^2\theta$

$*$ $\cot^2\theta + 1 = \mathrm{cosec}\,^2\theta$

- $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$*$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

- $\sin(A+B) = \sin A \cos B + \sin B \cos A$

∗ $\sin(A-B) = \sin A \cos B - \sin B \cos A$

∗ $\sin 2A = 2 \sin A \cos A$

∗ $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

∗ if $t = \tan \frac{\theta}{2}$, then $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\tan\theta = \frac{2t}{1-t^2}$

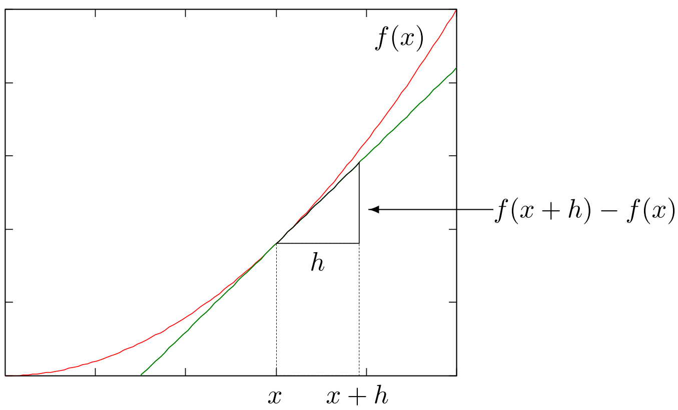
4 Calculus

4.1 Differentiation

4.1.1 Definition

The derivative of a function, $f(x)$, represents the gradient (rate of change) of that function. It is defined by:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



The above definition of differentiation and will be covered more fully in lectures.

4.1.2 Derivatives and Rules

Below are some standard results which should be known by heart:

- $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$

- $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$

Below are some standard rules which allow compound functions to be differentiated.

Linearity rule: $\frac{\mathrm{d}}{\mathrm{d}x}(af(x) + bg(x)) = a\frac{\mathrm{d}f}{\mathrm{d}x} + b\frac{\mathrm{d}g}{\mathrm{d}x}$

Chain rule: $\frac{\mathrm{d}}{\mathrm{d}x}f(u) = \frac{\mathrm{d}f}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

Product rule: $\frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$

Quotient rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g\frac{\mathrm{d}f}{\mathrm{d}x} - f\frac{\mathrm{d}g}{\mathrm{d}x}}{g^2}$

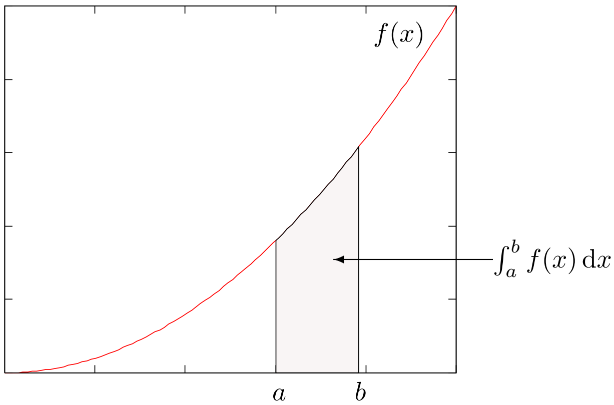
Below are some results which can be worked out by by using standard rules and standard results. The results don't have to be known by heart but you should understand how to reproduce them.

- $\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x$ [Tip: rewrite $a^x = (e^{\ln a})^x$]
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\mathrm{cosec}\,^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \sec(\arcsin x)$
  [Tip: rewrite $y = \arcsin x$ as $\sin y = x$ before differentiating]
- $\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\mathrm{cosec}\,(\arccos x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \cos^2(\arctan x)$

4.2 Integration

4.2.1 Definition

The integral of a function between two bounds measures the area under the function:



Integration is the inverse of differentiation and vice-versa, that is (for a smooth region of the function $f(x)$):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(u)\,\mathrm{d}u\right) = f(x) \quad \text{and} \quad \int_a^x \left(\frac{\mathrm{d}}{\mathrm{d}u} f(u)\right)\mathrm{d}u = f(x) - f(a)$$

This is the *fundamental theorem of calculus* and can be proved using the definition of the derivative and integral (above).

4.2.2 Integrals and Rules

Indefinite integrals require a constant of integration, C:

- $\displaystyle\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad : n \neq -1$
- $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$
- $\displaystyle\int e^x\,\mathrm{d}x = e^x + C$
- $\displaystyle\int \sin x\,\mathrm{d}x = -\cos x + C$
- $\displaystyle\int \cos x\,\mathrm{d}x = \sin x + C$

As with differentiation, their is a linearity rule:

Linearity rule: $\displaystyle\int af(x) + bg(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x + b\int g(x)\,\mathrm{d}x$

Some useful integration tricks include:

- $\displaystyle\int f'(x)\,\mathrm{d}x = f(x) + C$
- $\displaystyle\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$
- $\displaystyle\int f'(x)e^{f(x)}\,\mathrm{d}x = e^{f(x)} + C$

where $f'(x) \equiv \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}$.

The last of these two tricks are special cases of the substitution technique where $u = f(x)$ leads to the informal substitution $\mathrm{d}u \equiv f'(x)\mathrm{d}x$. For example, for the integral $\int e^{ax+b}\,\mathrm{d}x$ can be reduced to $\int \frac{1}{a}e^{u}\,\mathrm{d}u$ by using the substitution $u = ax + b$ and thus $\mathrm{d}u \equiv a\,\mathrm{d}x$ which in this case we use as $\mathrm{d}x \equiv \frac{1}{a}\mathrm{d}u$.

Finally, we have **integration by parts** which can be useful (but not always successful) for integrating products of functions:

Indefinite $\displaystyle\int f(x)g(x)\,\mathrm{d}x = f(x)\int g(x)\,\mathrm{d}x - \int f'(x)\left(\int g(x)\,\mathrm{d}x\right)\mathrm{d}x$

Definite $\displaystyle\int_a^b f(x)g(x)\,\mathrm{d}x = \left[f(x)\int_c^x g(u)\,\mathrm{d}u\right]_a^b - \int_a^b f'(x)\left(\int_c^x g(u)\,\mathrm{d}u\right)\mathrm{d}x$