Probabilistic Models for Data-Efficient Reinforcement Learning

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Creative Machine Learning

Karlsruhe Institute of Technology

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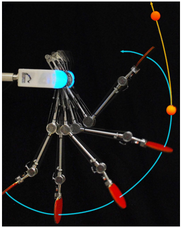
Autonomous Robots: Key Challenges

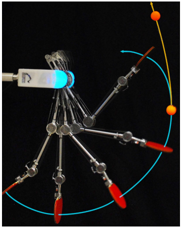
- Three key challenges in autonomous robots:
  Modeling. Predicting. Decision making.



Robotics

Autonomous Robots: Key Challenges

- Three key challenges in autonomous robots:
  Modeling. Predicting. Decision making.
- No human in the loop ⏩ "Learn" from data
- Automatically extract information
- Data-efficient (fast) learning
- Uncertainty: sensor noise, unknown processes, limited knowledge, ...



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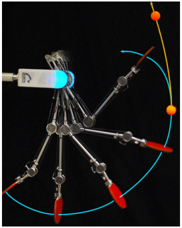
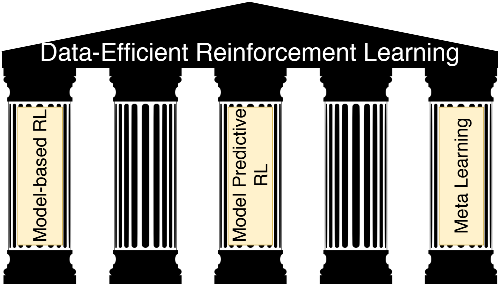
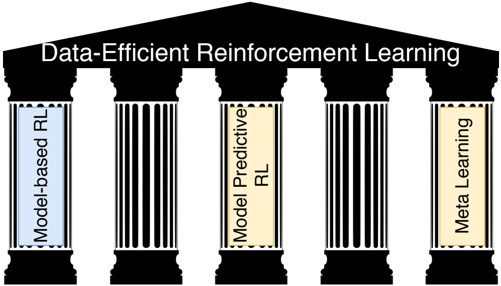


Robotics

⏩ **Reinforcement learning**
subject to data efficiency



1. **Model-based RL**
   - Data-efficient decision making
2. **Model predictive RL**
   - Speed up learning further by online planning
3. **Meta learning** using latent variables
   - Generalize knowledge to new situations


Data-Efficient Reinforcement Learning
Model-based RL
Model Predictive RL
Meta Learning

Reinforcement Learning and Optimal Control

$$\boldsymbol{x}_{t+1} = f(\boldsymbol{x}_t, \boldsymbol{u}_t) + \boldsymbol{w}\,, \qquad \boldsymbol{u}_t = \pi(\boldsymbol{x}_t, \boldsymbol{\theta})$$

State Control Policy Policy parameters

Transition function

Reinforcement Learning and Optimal Control

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State

Transition function

Control

Policy

Policy parameters

Objective (Controller Learning)

Find policy parameters $\boldsymbol{\theta}^*$ that minimize the expected long-term cost

$$J(\boldsymbol{\theta}) = \sum_{t=1}^{T} \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]\,, \qquad p(\boldsymbol{x}_0) = \mathcal{N}\big(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\big)\,.$$

Instantaneous cost $c(\boldsymbol{x}_t)$, e.g., $\|\boldsymbol{x}_t - \boldsymbol{x}_{\text{target}}\|^2$

- Typical objective in optimal control and reinforcement learning (Bertsekas, 2005; Sutton & Barto, 1998)

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   ▸▸ System identification

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

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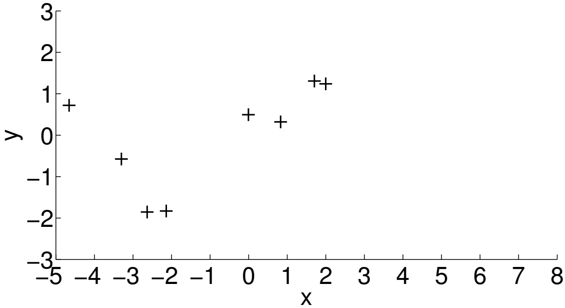
Model learning problem: Find a function $f : x \mapsto f(x) = y$



Observed function values

Model Learning (System Identification)

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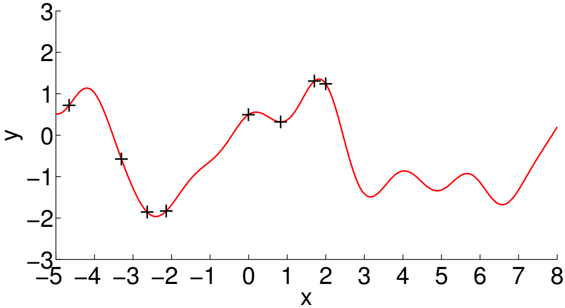

Plausible model

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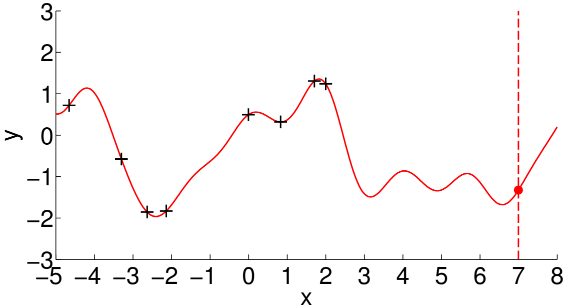

Plausible model

Predictions? Decision Making?

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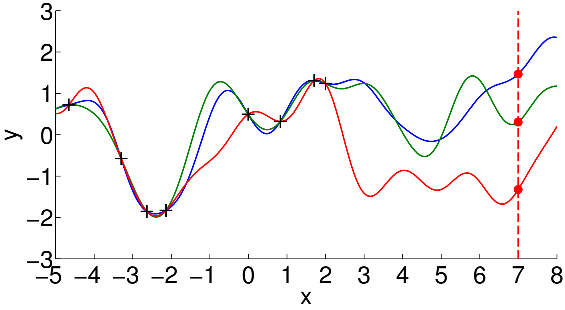

More plausible models

Predictions? Decision Making? Model Errors!

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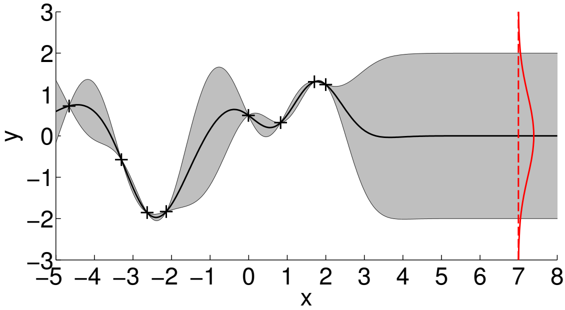

Distribution over plausible functions

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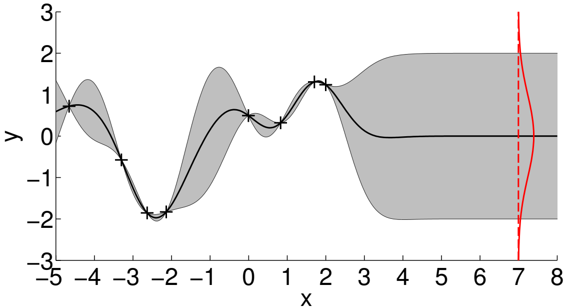

Distribution over plausible functions

- Express uncertainty about the underlying function to be robust to model errors
- Gaussian process for model learning (Rasmussen & Williams, 2006)

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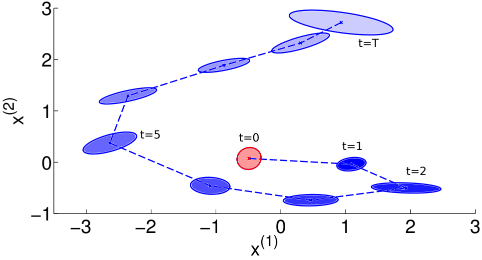
Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Long-Term Predictions



- Iteratively compute $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

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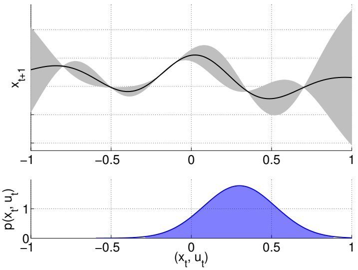

- Iteratively compute $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

$$\underbrace{p(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, \boldsymbol{u}_t)}_{\text{GP prediction}} \; \underbrace{p(\boldsymbol{x}_t, \boldsymbol{u}_t|\boldsymbol{\theta})}_{\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})}$$

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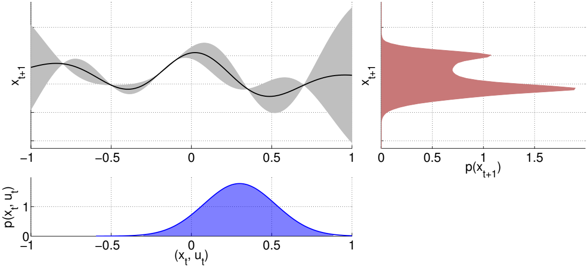

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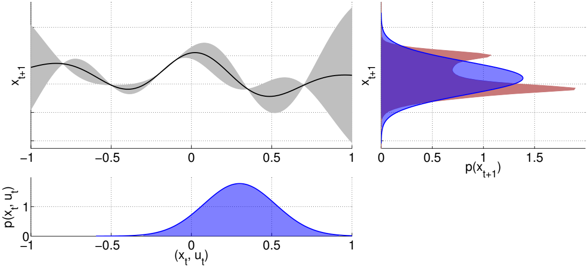
Long-Term Predictions



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▶▶ GP moment matching
(Girard et al., 2002; Quiñonero-Candela et al., 2003)

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   - Find parameters $\boldsymbol{\theta}$ that minimize $J(\boldsymbol{\theta})$
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Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

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- Compute

$$\mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}] = \int c(\boldsymbol{x}_t)\mathcal{N}\big(\boldsymbol{x}_t \,|\, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\big) d\boldsymbol{x}_t \,, \quad t = 1, \ldots, T \,,$$

and sum them up to obtain $J(\boldsymbol{\theta})$

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  and sum them up to obtain $J(\boldsymbol{\theta})$
- Analytically compute gradient $\mathrm{d}J(\boldsymbol{\theta})/\mathrm{d}\boldsymbol{\theta}$
- Standard gradient-based optimizer (e.g., BFGS) to find $\boldsymbol{\theta}^*$

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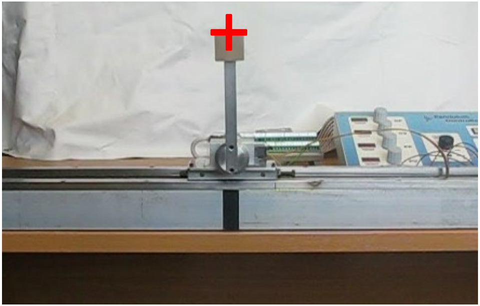
Standard Benchmark: Cart-Pole Swing-up



- Swing up and balance a freely swinging pendulum on a cart
- No knowledge about nonlinear dynamics ⏩ Learn from scratch
- Cost function $c(\boldsymbol{x}) = 1 - \exp(-\|\boldsymbol{x} - \boldsymbol{x}_{\text{target}}\|^2)$

- Code: https://github.com/ICL-SML/pilco-matlab

Deisenroth & Rasmussen (ICML, 2011): *PILCO: A Model-based and Data-efficient Approach to Policy Search*

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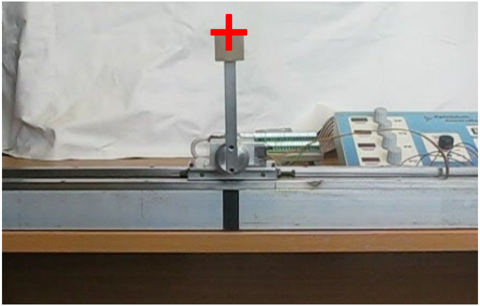

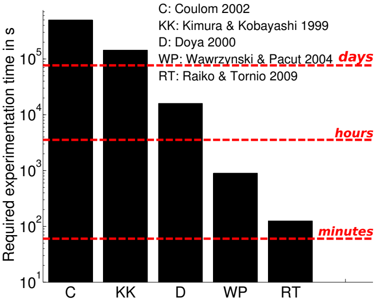

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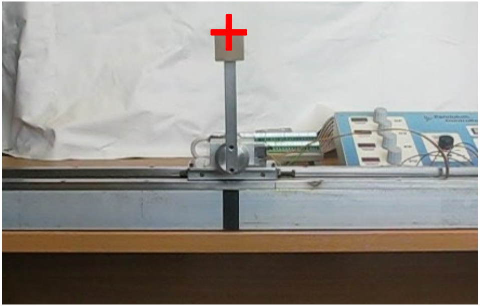

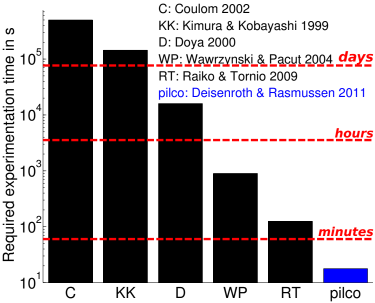

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- No knowledge about nonlinear dynamics ⏩ Learn from scratch
- Cost function $c(\boldsymbol{x}) = 1 - \exp(-\|\boldsymbol{x} - \boldsymbol{x}_{\text{target}}\|^2)$
- **Unprecedented learning speed** compared to state-of-the-art
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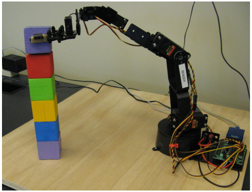
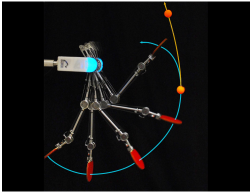
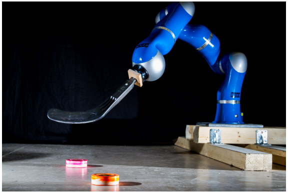
Wide Applicability



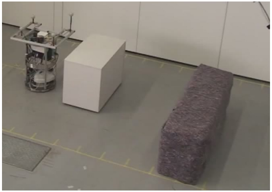
with D Fox



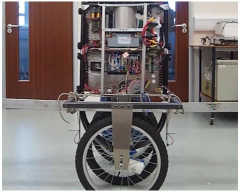
with P Englert, A Paraschos, J Peters



with A Kupcsik, J Peters, G Neumann



B Bischoff (Bosch), ESANN 2013



A McHutchon (U Cambridge)

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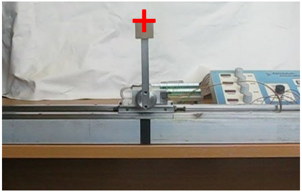
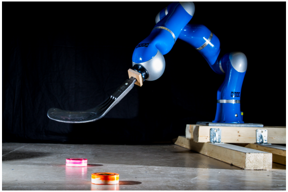
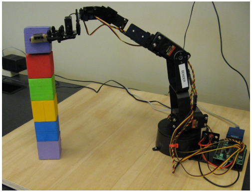
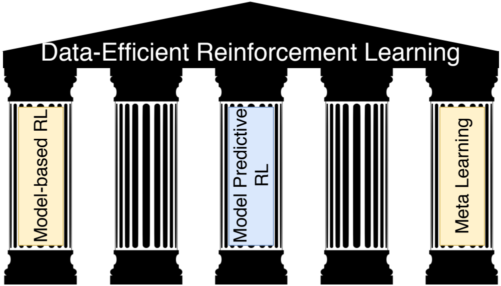
▶▶ Application to a wide range of robotic systems

Deisenroth et al. (RSS, 2011): *Learning to Control a Low-Cost Manipulator using Data-efficient Reinforcement Learning*
Englert et al. (ICRA, 2013): *Model-based Imitation Learning by Probabilistic Trajectory Matching*
Deisenroth et al. (ICRA, 2014): *Multi-Task Policy Search for Robotics*
Kupcsik et al. (AIJ, 2017): *Model-based Contextual Policy Search for Data-Efficient Generalization of Robot Skills*

Summary (1)

- In robotics, data-efficient learning is critical
- Probabilistic, model-based RL approach
  - Reduce model bias
  - Unprecedented learning speed
  - Wide applicability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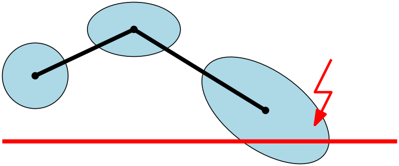
Safe Exploration



- Deal with real-world safety constraints (states/controls)
- Use probabilistic model to predict whether state constraints are violated (e.g., Sui et al., 2015; Berkenkamp et al., 2017)
- Adjust policy if necessary (during policy learning)

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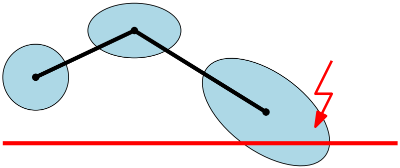

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▸▸ Safe exploration within an MPC-based RL setting

▸▸ Optimize control signals $\boldsymbol{u}_t$ directly (no policy parameters)

Approach

- Idea: Optimize control signals directly (instead of policy parameters)
- Few parameters to optimize ⏩ Low-dimensional search space
- Open-loop control
  ⏩ No chance of success (with minor model inaccuracies)

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- Few parameters to optimize ⏩ Low-dimensional search space
- Open-loop control
  ⏩ No chance of success (with minor model inaccuracies)
- Model predictive control (MPC) turns this into a closed-loop control approach
- Use this within a trial-and-error RL setting

Probabilistic MPC in RL

- Learned GP model for transition dynamics
- Repeat (while executing the policy):
  1. In current state $\boldsymbol{x}_t$, determine optimal control sequence $\boldsymbol{u}_0^*, \ldots, \boldsymbol{u}_{H-1}^*$
  2. Apply first control $\boldsymbol{u}_0^*$ in state $\boldsymbol{x}_t$
  3. Transition to next state $\boldsymbol{x}_{t+1}$
  4. Update GP transition model

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Theoretical Results

- Uncertainty propagation is deterministic (GP moment matching)
  - Re-formulate system dynamics:

$$\boldsymbol{z}_{t+1} = f_{MM}(\boldsymbol{z}_t, \boldsymbol{u}_t)$$

$$\boldsymbol{z}_t = \{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$$ ▸▸ Collects moments

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- Deterministic system function that propagates moments
- Lipschitz continuity (under mild assumptions) implies that we can apply Pontryagin's Minimum Principle
  ⏩ Principled treatment of control constraints

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Theoretical Results

- Uncertainty propagation is deterministic (GP moment matching)
  - Re-formulate system dynamics:

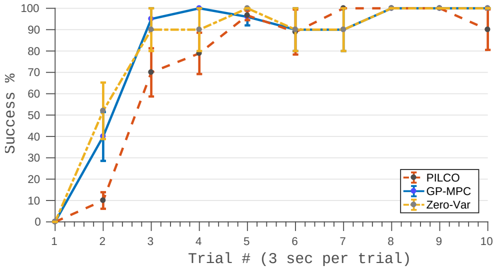
$$\boldsymbol{z}_{t+1} = f_{MM}(\boldsymbol{z}_t, \boldsymbol{u}_t)$$

$$\boldsymbol{z}_t = \{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$$ ▸▸ Collects moments

- Deterministic system function that propagates moments
- Lipschitz continuity (under mild assumptions) implies that we can apply Pontryagin's Minimum Principle
  - Principled treatment of control constraints
- Use predictive uncertainty to check violation of state constraints

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Learning Speed (Cart Pole)



- Zero-Var: Only use the mean of the GP, discard variances for long-term predictions

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

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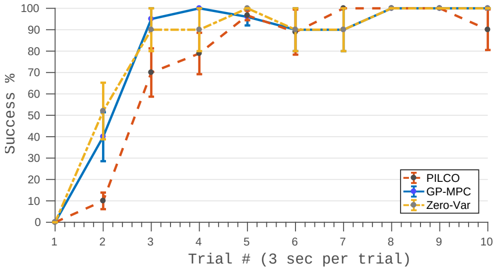

- Zero-Var: Only use the mean of the GP, discard variances for long-term predictions
- MPC: Increased data efficiency (40% less experience required than PILCO)

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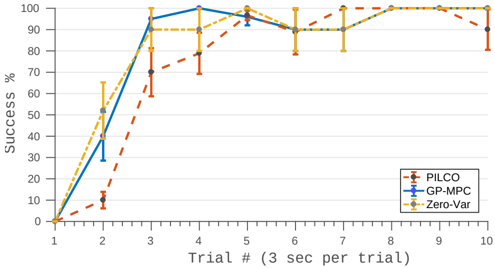
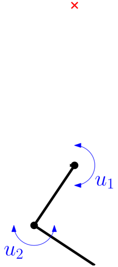
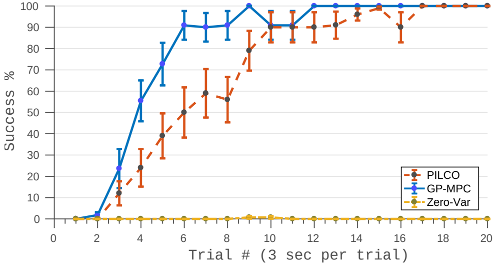


- Zero-Var: Only use the mean of the GP, discard variances for long-term predictions
- MPC: Increased data efficiency (40% less experience required than PILCO)
- MPC more robust to model inaccuracies than a parametrized feedback controller

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Learning Speed (Double Pendulum)





- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
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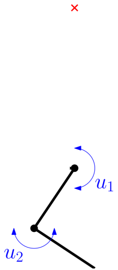
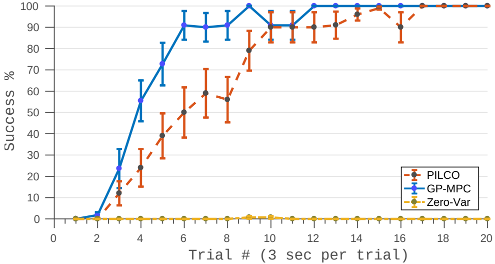
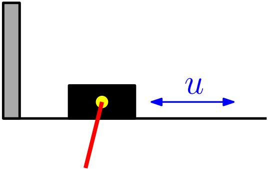


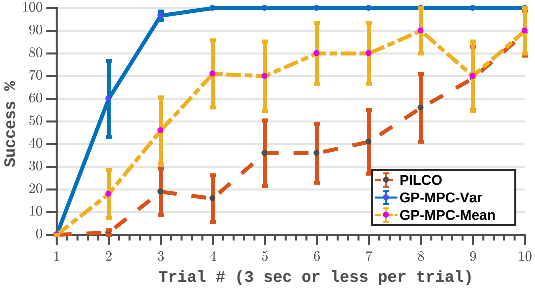


- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
  - Gets stuck in local optimum near start state
  - Insufficient exploration due to lack of uncertainty propagation
- Although MPC is fairly robust to model inaccuracies we cannot get away without uncertainty propagation

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Safety Constraints (Cart Pole)



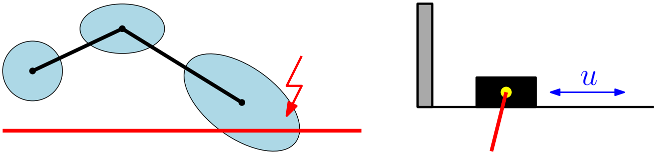


PILCO	16/100	constraint violations
GP-MPC-Mean	21/100	constraint violations
GP-MPC-Var	3/100	constraint violations

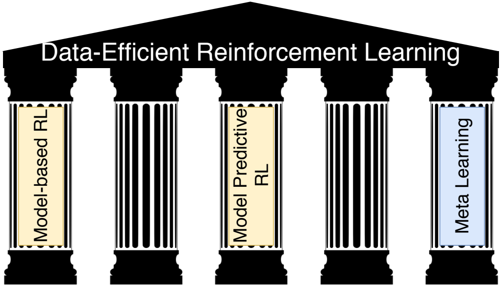
Propagating model uncertainty important for safety

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Summary (2)



- Probabilistic prediction models for safe exploration
- Uncertainty propagation can be used to reduce violation of safety constraints
- MPC framework increases robustness to model errors
  ▶▶ Increased data efficiency


Data-Efficient Reinforcement Learning
Model-based RL
Model Predictive RL
Meta Learning

Meta Learning

Generalize knowledge from known tasks to new (related) tasks

Meta Learning

Meta Learning

Generalize knowledge from known tasks to new (related) tasks

- Different robot configurations (link lengths, weights, ...)
- Re-use experience gathered so far generalize learning to new dynamics that are similar
  ▸▸ Accelerated learning

Approach

- Separate global and task-specific properties
- Shared global parameters describe general dynamics
- Describe task-specific (local) configurations with latent variable

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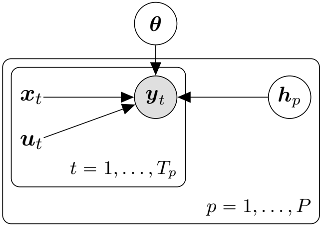
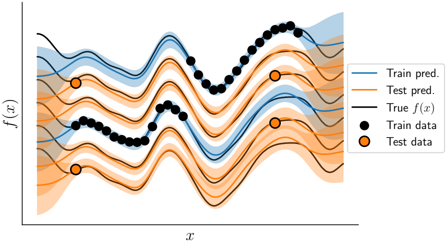
- Separate global and task-specific properties
- Shared global parameters describe general dynamics
- Describe task-specific (local) configurations with latent variable
- Online variational inference of (unseen) configurations

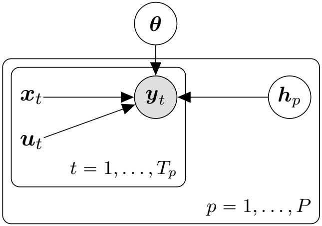
Meta Model Learning with Latent Variables

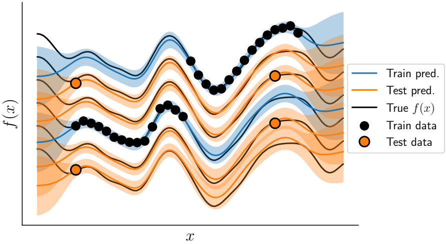




$$y_t = f(x_t, u_t, h_p; \theta)$$

Sæmundsson et al. (UAI, 2018): *Meta Reinforcement Learning with Latent Variable Gaussian Processes*





$$\boldsymbol{y}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{h}_p; \boldsymbol{\theta})$$

- GP captures global (shared) properties of the dynamics

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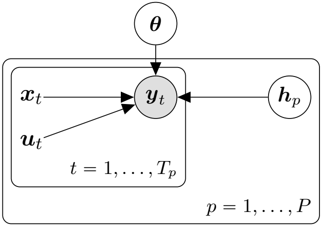

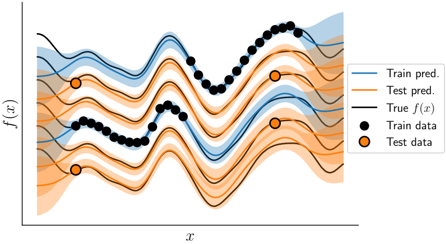

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- Latent variable $\boldsymbol{h}_p$ describes task-specific properties
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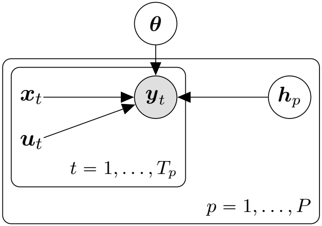
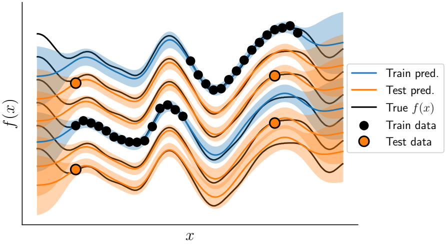
Meta Model Learning with Latent Variables



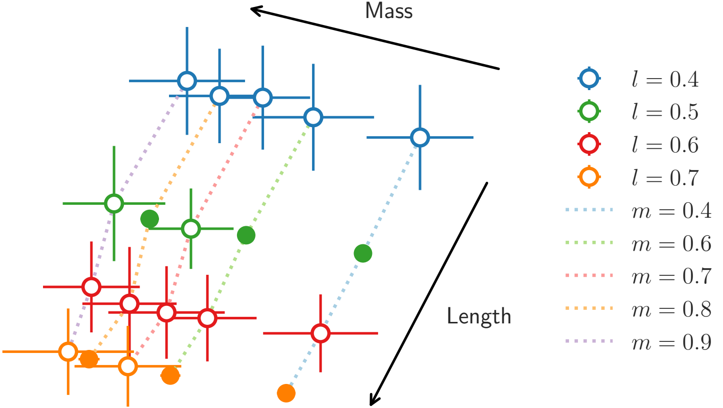


$$y_t = f(x_t, u_t, h_p; \theta)$$

- GP captures global (shared) properties of the dynamics
- Latent variable h_p describes task-specific properties
  ▸▸ Variational inference to find a posterior on latent configuration
- Fast online inference of new configurations (no model re-training required)

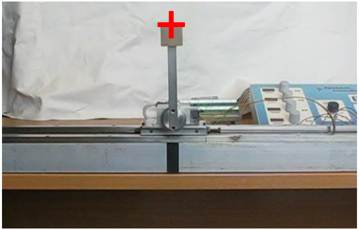
Sæmundsson et al. (UAI, 2018): *Meta Reinforcement Learning with Latent Variable Gaussian Processes*

Latent Embeddings



- Latent variable $\boldsymbol{h}$ encodes length l and mass m of the cart pole
- 6 training tasks, 14 held-out test tasks

Sæmundsson et al. (UAI, 2018): *Meta Reinforcement Learning with Latent Variable Gaussian Processes*

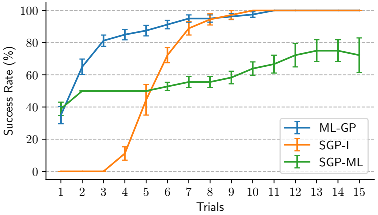


- Pre-trained on 6 training configurations until solved

Model	Training (s)	Description
Independent	16.1 ± 0.4	Independent GP-MPC
Aggregated	23.7 ± 1.4	Aggregated experience (no latents)
Meta learning	$\mathbf{15.1 \pm 0.5}$	Aggregated experience (with latents)

Meta learning can help speeding up RL

Sæmundsson et al. (UAI, 2018): *Meta Reinforcement Learning with Latent Variable Gaussian Processes*

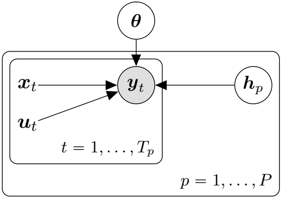


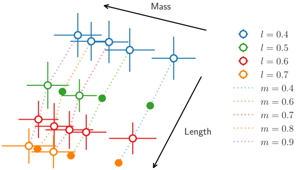
- Few-shot generalization on 4 unseen configurations
- Success: solve all 10 (6 training + 4 test) tasks
- Meta learning: blue
- Independent (GP-MPC): orange
- Aggregated experience model (no latents): green

Meta RL generalizes well to unseen tasks

Sæmundsson et al. (UAI, 2018): *Meta Reinforcement Learning with Latent Variable Gaussian Processes*

Summary (3)

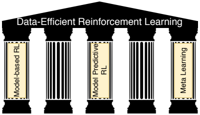


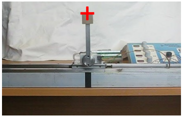


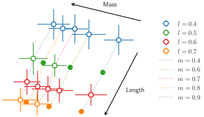
- Generalize knowledge from known situations to unseen ones
  ▶▶ **Few-shot learning**
- Latent variable can be used to infer task similarities
- Significant speed-up in model learning and model-based RL

Wrap-Up

UCL

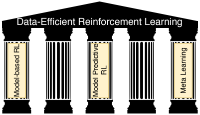


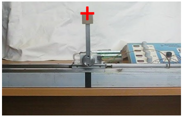


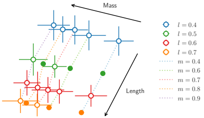


- **Data efficiency** is a practical challenge for autonomous robots
- Three pillars of data-efficient reinforcement learning for autonomous robots
  1. Model-based reinforcement learning with learned probabilistic models for fast learning from scratch
  2. Model predictive reinforcement learning with learned dynamics models accelerates learning and allow for safe exploration
  3. Meta learning using latent variables to generalize knowledge to new situations
- **Key to success:** Probabilistic modeling and Bayesian inference

Wrap-Up







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Thank you for your attention

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Controller Parametrization

- Controller:

$$\tilde{\pi}(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_{k=1}^{K} w_k \exp\big(-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_k)^\top \boldsymbol{\Lambda}(\boldsymbol{x} - \boldsymbol{\mu}_k)\big)$$

$$u = \pi(\boldsymbol{x}, \boldsymbol{\theta}) = u_{\max}\sigma(\tilde{\pi}(\boldsymbol{x}, \boldsymbol{\theta})) \in [-u_{\max}, u_{\max}]\,,$$

where σ is a squashing function.

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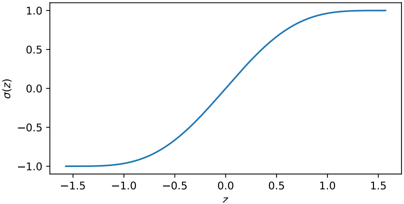
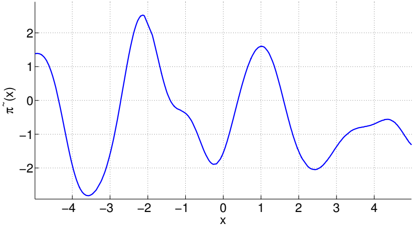
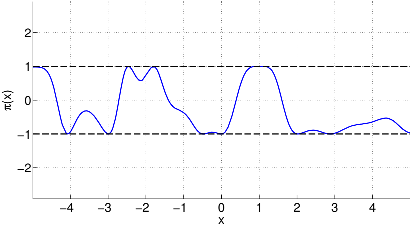
- Squashing function:

$$\sigma(z) = \frac{9}{8}\sin(z) + \frac{1}{8}\sin(3z)$$

Squashing Function

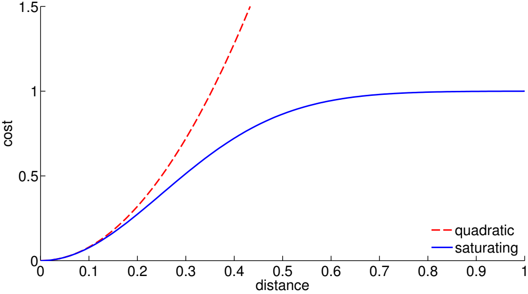






Cost Functions

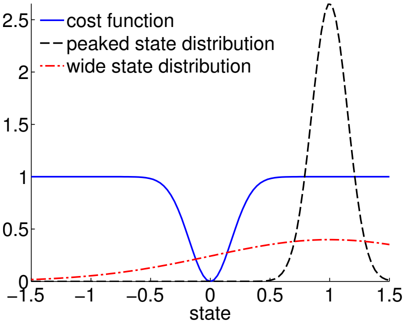
- Quadratic cost $c(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{x}_{\text{target}})^\top \boldsymbol{W} (\boldsymbol{x} - \boldsymbol{x}_{\text{target}})$
- Saturating cost $c(\boldsymbol{x}) = 1 - \exp\left(-(\boldsymbol{x} - \boldsymbol{x}_{\text{target}})^\top \boldsymbol{W} (\boldsymbol{x} - \boldsymbol{x}_{\text{target}})\right)$



- Quadratic cost pays a lot of attention to states "far away"
  - ▸▸ Bad idea for exploration

Natural Exploration with the Saturating Cost

Task: Minimize $\mathbb{E}[c(\boldsymbol{x}_t)]$



- In the early stages of learning, state predictions are expected to be far away from the target

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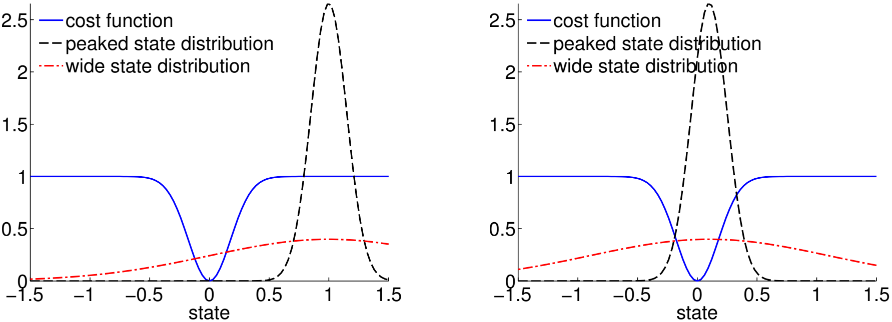

- In the early stages of learning, state predictions are expected to be far away from the target ⏩ Exploration favored

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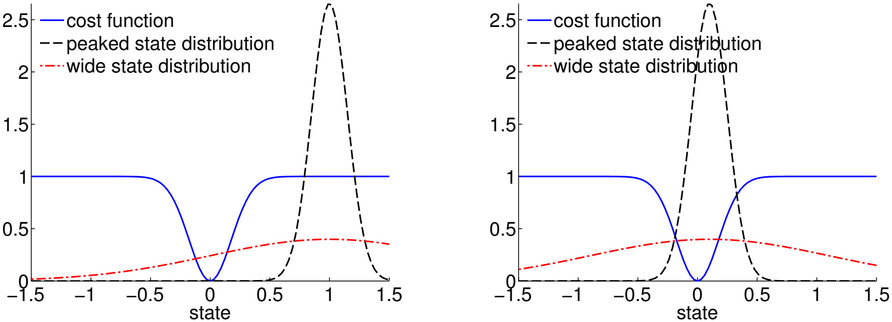

- In the early stages of learning, state predictions are expected to be far away from the target ▸▸ Exploration favored
- In the final stages of learning, state predictions are expected to be close to the target

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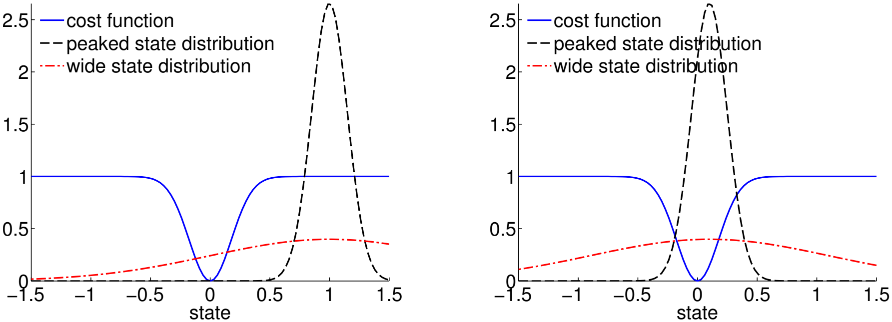

- In the early stages of learning, state predictions are expected to be far away from the target ⏩ Exploration favored
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Natural Exploration with the Saturating Cost

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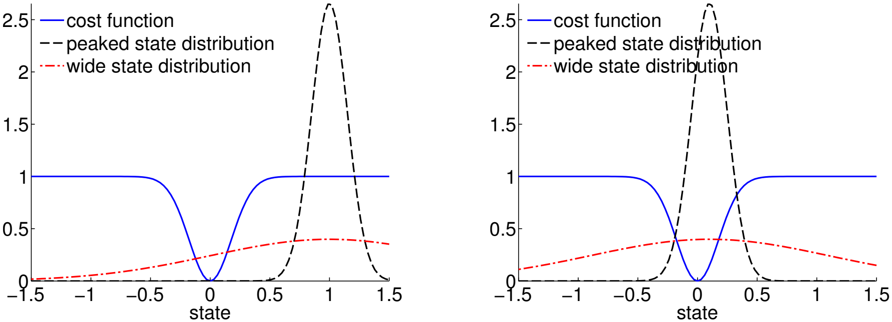

- In the early stages of learning, state predictions are expected to be far away from the target ▶▶ Exploration favored
- In the final stages of learning, state predictions are expected to be close to the target ▶▶ Exploitation favored

▶▶ Bayesian treatment: Natural exploration/exploitation trade-off

GP Moment Matching: Some Details

$$f \sim GP(0, k)\,, \quad \text{Training data: } \boldsymbol{X}, \boldsymbol{y}$$
$$\boldsymbol{x}_* \sim \mathcal{N}\big(\boldsymbol{\mu}, \boldsymbol{\Sigma}\big)$$

- Compute $\mathbb{E}[f(\boldsymbol{x}_*)]$

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&= \mathbb{E}_{\boldsymbol{x}_*}\big[k(\boldsymbol{x}_*, \boldsymbol{X})(\boldsymbol{K} + \sigma_n^2 \boldsymbol{I})^{-1}\boldsymbol{y}\big] \\
&= \boldsymbol{\beta}^\top \int k(\boldsymbol{X}, \boldsymbol{x}_*)\mathcal{N}\big(\boldsymbol{x}_* \,|\, \boldsymbol{\mu},\, \boldsymbol{\Sigma}\big) d\boldsymbol{x}_*
\end{aligned}$$

$\boldsymbol{\beta} := (\boldsymbol{K} + \sigma_n^2 \boldsymbol{I})^{-1}\boldsymbol{y}$ ▶▶ independent of $\boldsymbol{x}_*$

GP Moment Matching: Some Details

$$f \sim GP(0, k)\,, \quad \text{Training data: } \boldsymbol{X}, \boldsymbol{y}$$
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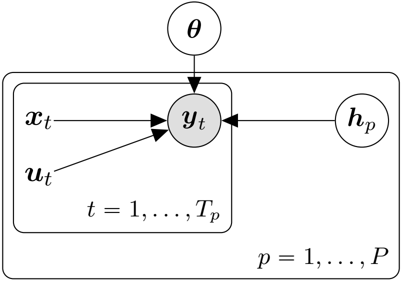
\end{aligned}$$

$$\boldsymbol{\beta} := (\boldsymbol{K} + \sigma_n^2\boldsymbol{I})^{-1}\boldsymbol{y} \quad \blacktriangleright\!\blacktriangleright \text{ independent of } \boldsymbol{x}_*$$

- If k is a Gaussian (squared exponential) kernel, this integral can be solved analytically
- Variance of $f(\boldsymbol{x}_*)$ can be computed similarly

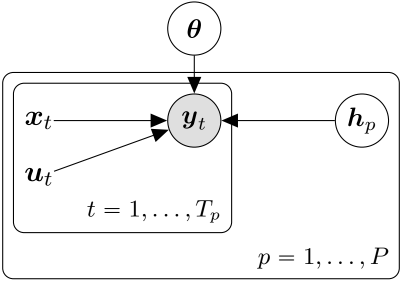
Meta Learning Model



$$\boldsymbol{f}(\cdot) \sim GP$$

$$p(\boldsymbol{H}) = \prod_p p(\boldsymbol{h}_p)\,, \quad p(\boldsymbol{h}_p) = \mathcal{N}\big(\boldsymbol{0},\, \boldsymbol{I}\big)$$

Meta Learning Model



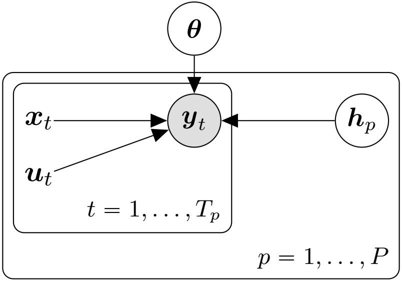
$$\boldsymbol{f}(\cdot) \sim GP$$

$$p(\boldsymbol{H}) = \prod_p p(\boldsymbol{h}_p)\,, \quad p(\boldsymbol{h}_p) = \mathcal{N}\big(\boldsymbol{0},\, \boldsymbol{I}\big)$$

$$p(\boldsymbol{Y}, \boldsymbol{H}, \boldsymbol{f}(\cdot)|\boldsymbol{X}, \boldsymbol{U}) = \prod\nolimits_{p=1}^{P} p(\boldsymbol{h}_p) \prod\nolimits_{t=1}^{T_p} p(\boldsymbol{y}_t|\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{h}_p, \boldsymbol{f}(\cdot))p(\boldsymbol{f}(\cdot))$$

$$\boldsymbol{y}_t = \boldsymbol{x}_{t+1} - \boldsymbol{x}_t$$

Variational Inference in Meta Learning Model



Mean-field variational family:

$$q(\boldsymbol{f}(\cdot), \boldsymbol{H}) = q(\boldsymbol{f}(\cdot))q(\boldsymbol{H})$$

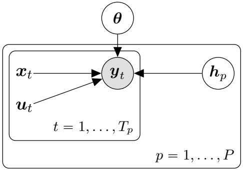
$$q(\boldsymbol{H}) = \prod_{p=1}^{P} \mathcal{N}(\boldsymbol{h}_p | \boldsymbol{n}_p, \boldsymbol{T}_p) \,,$$

$$q(\boldsymbol{f}(\cdot)) = \int p(\boldsymbol{f}(\cdot)|\boldsymbol{f}_Z)q(\boldsymbol{f}_Z)d\boldsymbol{f}_Z$$

▶▶ SV-GP (Titsias, 2009)

Evidence Lower Bound

$$ELBO = \mathbb{E}_{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big[\log \frac{p(\boldsymbol{Y},\boldsymbol{H},\boldsymbol{f}(\cdot)|\boldsymbol{X},\boldsymbol{U})}{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big]$$

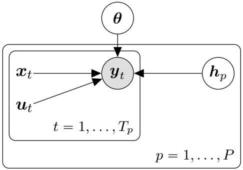


Evidence Lower Bound

$$ELBO = \mathbb{E}_{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big[\log \frac{p(\boldsymbol{Y},\boldsymbol{H},\boldsymbol{f}(\cdot)|\boldsymbol{X},\boldsymbol{U})}{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big]$$

$$= \sum_{p=1}^{P}\sum_{t=1}^{T_p}\mathbb{E}_{q(\boldsymbol{f}_t|\boldsymbol{x}_t,\boldsymbol{u}_t,\boldsymbol{h}_p)q(\boldsymbol{h}_p)}\big[\log p(\boldsymbol{y}_t|\boldsymbol{f}_t)\big]$$

$$- \mathrm{KL}(q(\boldsymbol{H})||p(\boldsymbol{H})) - \mathrm{KL}(q(\boldsymbol{f}(\cdot))||p(\boldsymbol{f}(\cdot)))$$



Evidence Lower Bound

$$
\begin{aligned}
ELBO &= \mathbb{E}_{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big[\log \frac{p(\boldsymbol{Y},\boldsymbol{H},\boldsymbol{f}(\cdot)|\boldsymbol{X},\boldsymbol{U})}{q(\boldsymbol{f}(\cdot),\boldsymbol{H})}\Big] \\
&= \sum_{p=1}^{P}\sum_{t=1}^{T_p}\mathbb{E}_{q(\boldsymbol{f}_t|\boldsymbol{x}_t,\boldsymbol{u}_t,\boldsymbol{h}_p)q(\boldsymbol{h}_p)}\big[\log p(\boldsymbol{y}_t|\boldsymbol{f}_t)\big] \\
&\quad - \mathrm{KL}(q(\boldsymbol{H})||p(\boldsymbol{H})) - \mathrm{KL}(q(\boldsymbol{f}(\cdot))||p(\boldsymbol{f}(\cdot))) \\
&= \sum_{p=1}^{P}\sum_{t=1}^{T_p}\overbrace{\mathbb{E}_{q(\boldsymbol{f}_t|\boldsymbol{x}_t,\boldsymbol{u}_t,\boldsymbol{h}_p)q(\boldsymbol{h}_p)}\big[\log p(\boldsymbol{y}_t|\boldsymbol{f}_t)\big]}^{\text{Monte Carlo estimate}} \\
&\quad \underbrace{-\mathrm{KL}(q(\boldsymbol{H})||p(\boldsymbol{H})) - \mathrm{KL}(q(\boldsymbol{F}_Z)||p(\boldsymbol{F}_Z))}_{\text{closed-form solution}}
\end{aligned}
$$