Controlling Mechanical Systems with Learned Models: A Machine Learning Approach

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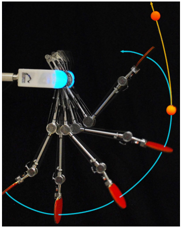
Autonomous Robots: Key Challenges

- Three key challenges in autonomous systems:

  Modeling. Predicting. Decision making.



Robotics

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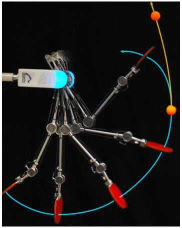
- No human in the loop ⏩ “Learn” from data
- Automatically extract information
- Data-efficient (fast) learning
- Uncertainty: sensor noise, unknown processes, limited knowledge, ...



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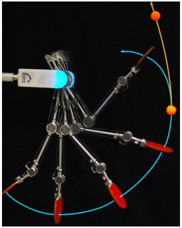

Robotics

⏩ **Reinforcement learning** subject to data efficiency

$$\boxed{\boldsymbol{x}_{t+1}} = f\left(\boxed{\boldsymbol{x}_t}, \boxed{\boldsymbol{u}_t}\right) + \boldsymbol{w}\,, \qquad \boxed{\boldsymbol{u}_t} = \boxed{\pi}\left(\boxed{\boldsymbol{x}_t}, \boxed{\boldsymbol{\theta}}\right)$$

State

Transition function

Control

Policy

Policy parameters

Reinforcement Learning

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State

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Policy parameters

Objective (Controller Learning)

Find policy parameters $\boldsymbol{\theta}^*$ that minimize the expected long-term cost

$$J(\boldsymbol{\theta}) = \sum_{t=1}^{T} \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]\,, \qquad p(\boldsymbol{x}_0) = \mathcal{N}\big(\boldsymbol{\mu}_0,\, \boldsymbol{\Sigma}_0\big)\,.$$

Instantaneous cost $c(\boldsymbol{x}_t)$, e.g., $\|\boldsymbol{x}_t - \boldsymbol{x}_{\text{target}}\|^2$

- Typical objective in optimal control and reinforcement learning (Bertsekas, 2005; Sutton & Barto, 1998)

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   - System identification

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

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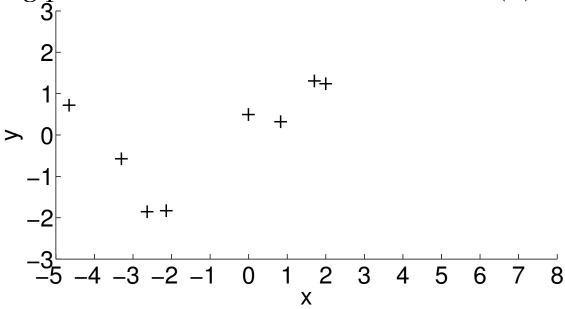
Model Learning (System Identification)

Model learning problem: Find a function $f : x \mapsto f(x) = y$



Observed function values

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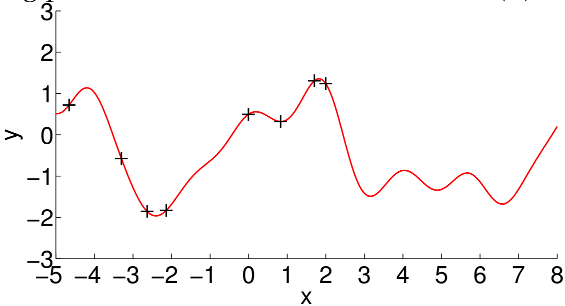

Plausible model

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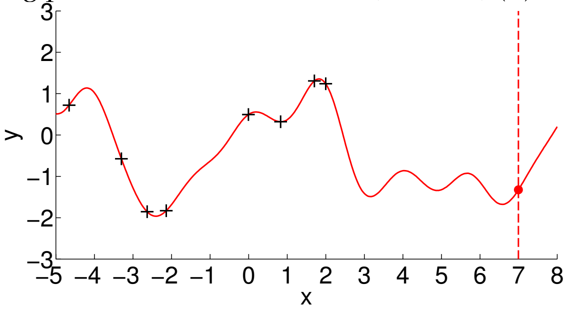

Plausible model

Predictions? Decision Making?

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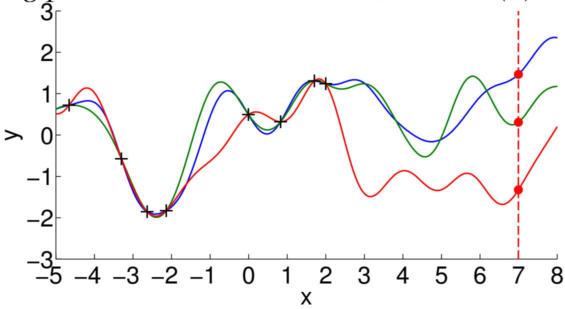

More plausible models

Predictions? Decision Making? Model Errors!

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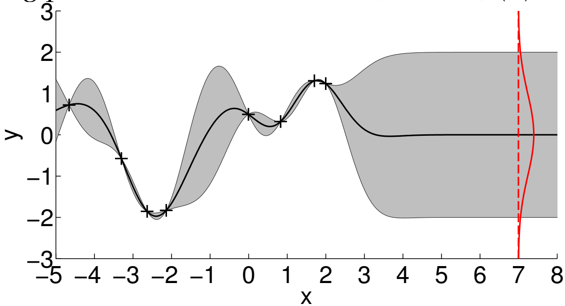

Distribution over plausible functions

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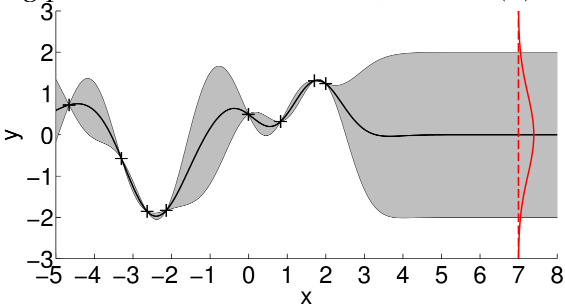

Distribution over plausible functions

- Express uncertainty about the underlying function to be robust to model errors
- Gaussian process for model learning (Rasmussen & Williams, 2006)

Introduction to Gaussian Processes

- Flexible Bayesian regression method
- Probability distribution over functions
- Fully specified by
  - Mean function m (average function)
  - Covariance function k (assumptions on structure)

$$k(\boldsymbol{x}_p, \boldsymbol{x}_q) = \mathrm{Cov}[f(\boldsymbol{x}_p), f(\boldsymbol{x}_q)]$$

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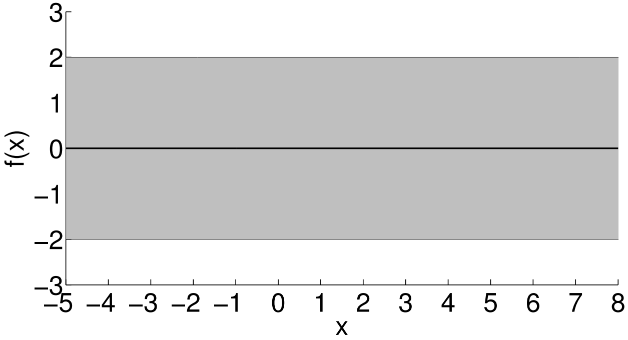
$$k(\boldsymbol{x}_p, \boldsymbol{x}_q) = \mathrm{Cov}[f(\boldsymbol{x}_p), f(\boldsymbol{x}_q)]$$

- Posterior predictive distribution at $\boldsymbol{x}_*$ is Gaussian (Bayes' theorem):

$$p(f(\boldsymbol{x}_*) | \boldsymbol{x}_*, \boldsymbol{X}, \boldsymbol{y}) = \mathcal{N}\big(f(\boldsymbol{x}_*) \,|\, m(\boldsymbol{x}_*),\, \sigma^2(\boldsymbol{x}_*)\big)$$

Test input Training data

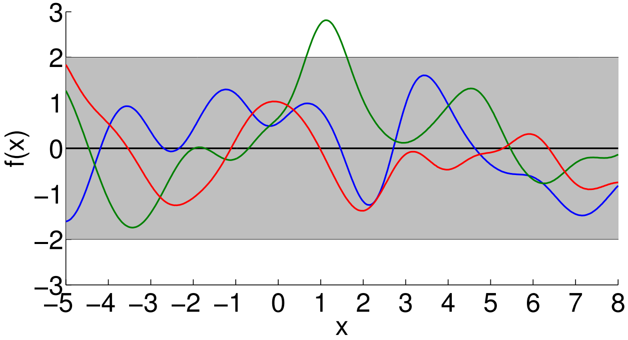


Prior belief about the function

Predictive (marginal) mean and variance:

$$\mathbb{E}[f(\boldsymbol{x}_*)|\boldsymbol{x}_*, \varnothing] = m(\boldsymbol{x}_*) = 0$$
$$\mathbb{V}[f(\boldsymbol{x}_*)|\boldsymbol{x}_*, \varnothing] = \sigma^2(\boldsymbol{x}_*) = k(\boldsymbol{x}_*, \boldsymbol{x}_*)$$

Intuitive Introduction to Gaussian Processes



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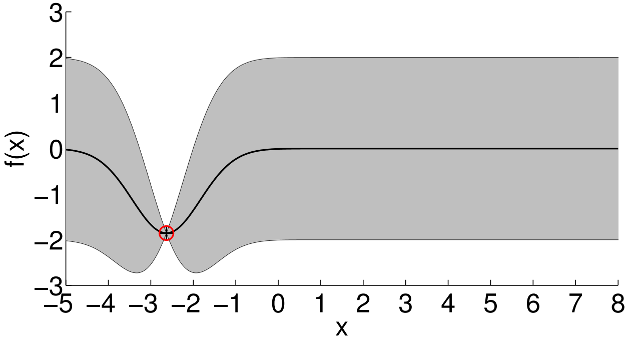

Posterior belief about the function

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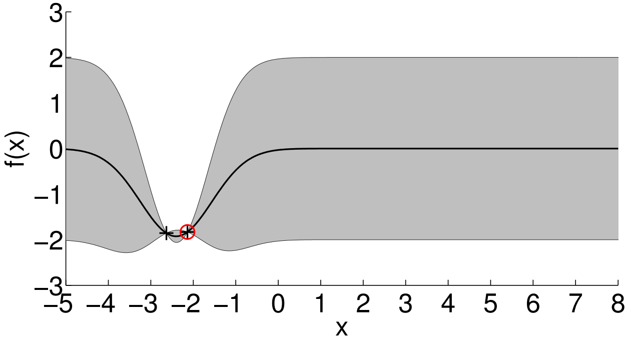

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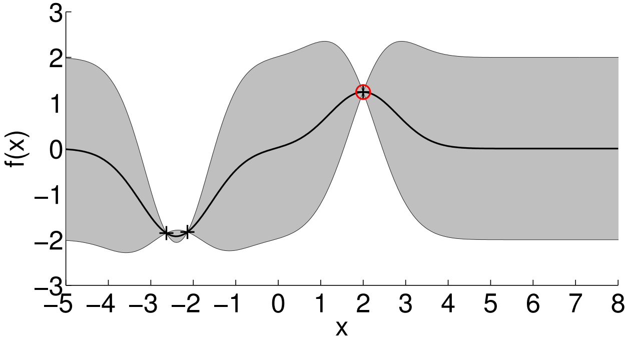

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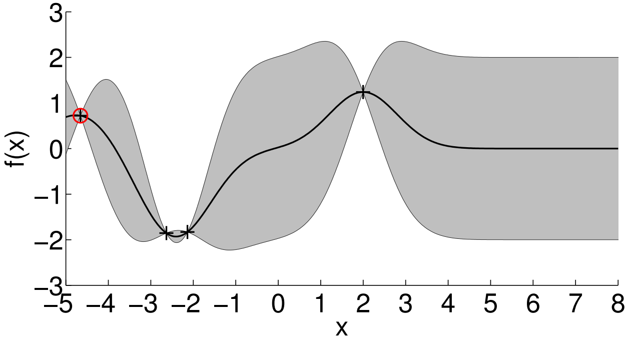

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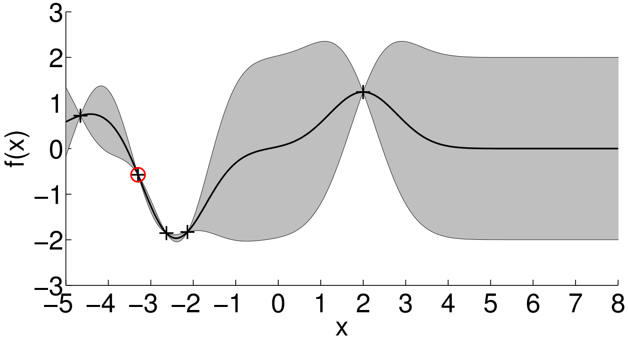

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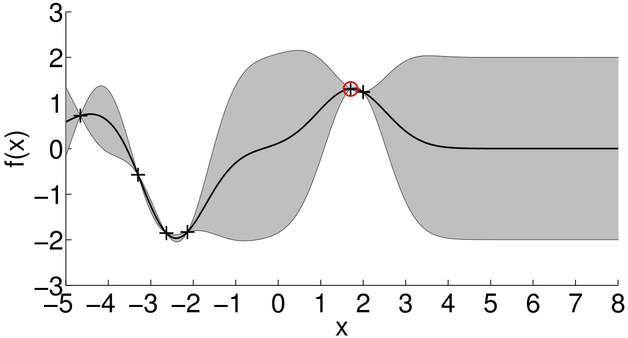

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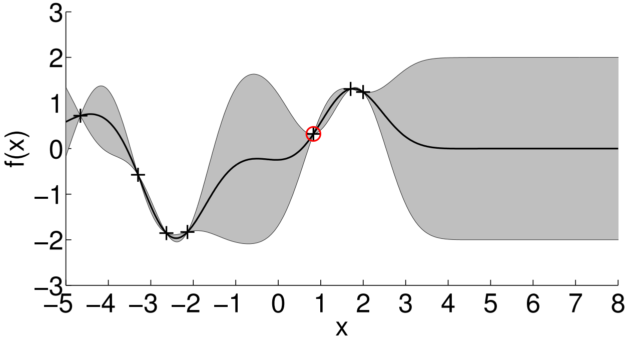

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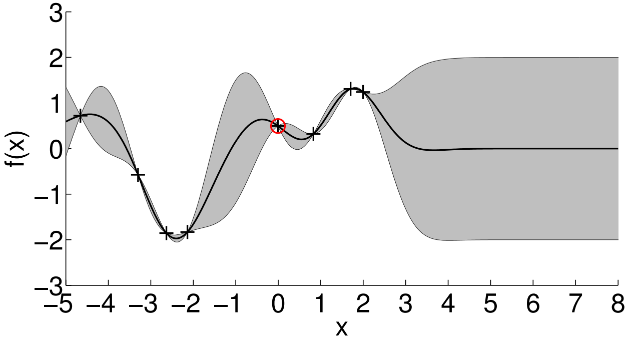

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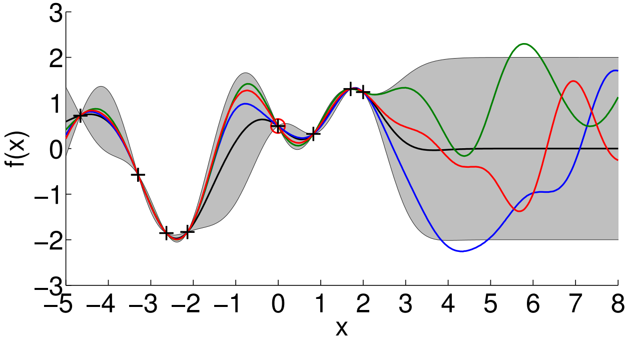

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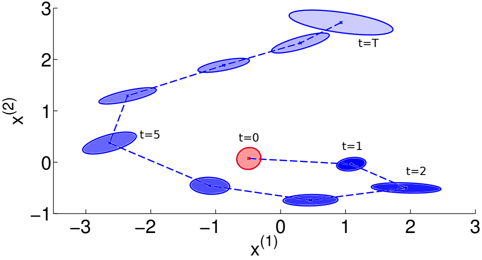
Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   ▸▸ System identification
2. **Compute long-term predictions** $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
3. Policy improvement
4. Apply controller

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Long-Term Predictions



- Iteratively compute $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

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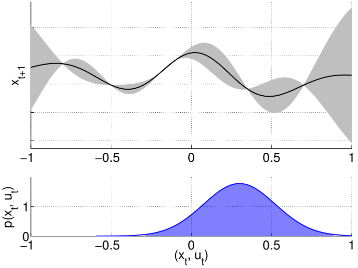

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$$\underbrace{p(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, \boldsymbol{u}_t)}_{\text{GP prediction}} \; \underbrace{p(\boldsymbol{x}_t, \boldsymbol{u}_t|\boldsymbol{\theta})}_{\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma})}$$

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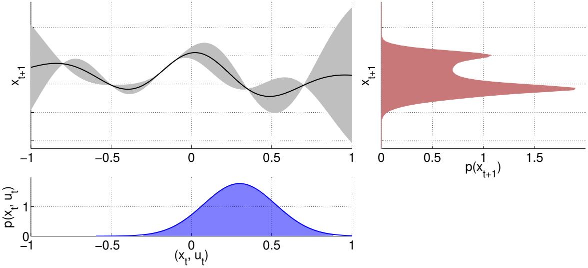

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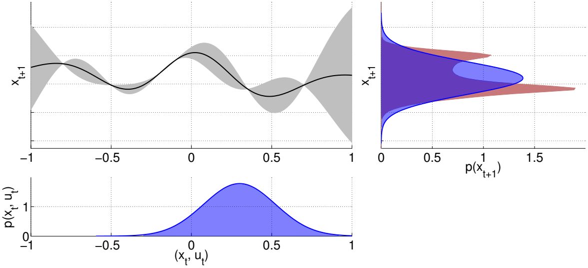
Long-Term Predictions



- Iteratively compute $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

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▶▶ GP moment matching
(Girard et al., 2002; Quiñonero-Candela et al., 2003)

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   - Compute expected long-term cost $J(\boldsymbol{\theta})$
   - Find parameters $\boldsymbol{\theta}$ that minimize $J(\boldsymbol{\theta})$
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Policy Improvement

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

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Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
- Compute

$$\mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}] = \int c(\boldsymbol{x}_t)\mathcal{N}\big(\boldsymbol{x}_t \,|\, \boldsymbol{\mu}_t,\, \boldsymbol{\Sigma}_t\big) d\boldsymbol{x}_t\,, \quad t = 1, \ldots, T\,,$$

and sum them up to obtain $J(\boldsymbol{\theta})$

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  and sum them up to obtain $J(\boldsymbol{\theta})$
- Analytically compute gradient $\mathrm{d}J(\boldsymbol{\theta})/\mathrm{d}\boldsymbol{\theta}$
- Standard gradient-based optimizer (e.g., BFGS) to find $\boldsymbol{\theta}^*$

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Fast Reinforcement Learning

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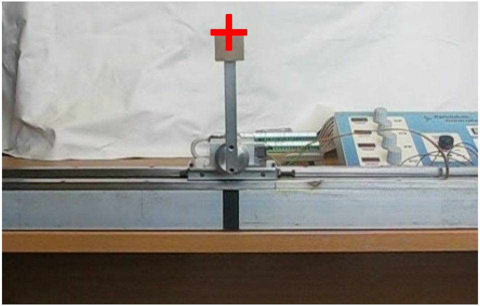
Standard Benchmark: Cart-Pole Swing-up



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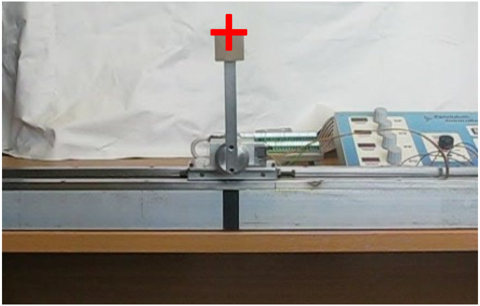

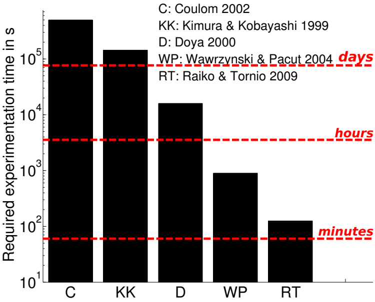

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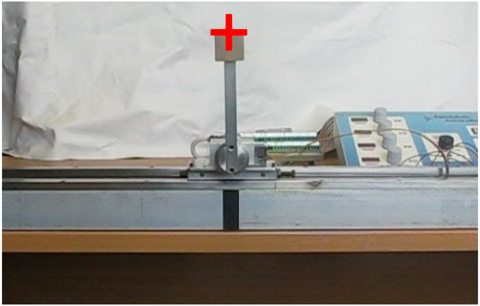

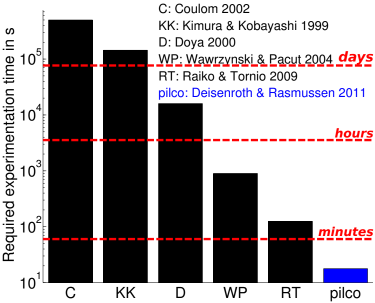

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- No knowledge about nonlinear dynamics ⏩ Learn from scratch
- Cost function $c(\boldsymbol{x}) = 1 - \exp(-\|\boldsymbol{x} - \boldsymbol{x}_{\text{target}}\|^2)$
- **Unprecedented learning speed** compared to state-of-the-art
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Probabilistic Model Essential?

DEMO

- Probabilistic model: GP
- Deterministic model: Mean function of GP (still nonparametric)

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Table: Average learning success with non-parametric transition models

	GP	"Deterministic" GP
Learning success	**94.52%**	**0%**

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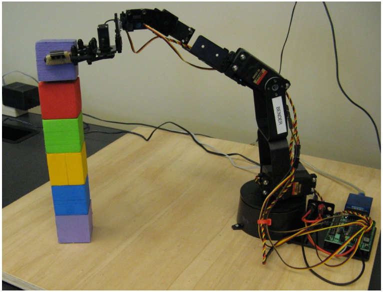
Reasons for failure of deterministic model:

- Model errors: Long-term predictions make absolutely no sense, and the predicted states are nowhere near the target
  - ▶▶ No gradient signal
- No automatic exploration (model and policy are deterministic)
  - ▶▶ Stochastic policy fixes this to some degree

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

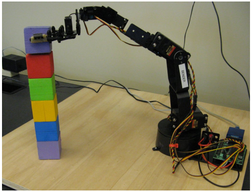
Learning to Control an Off-the-Shelf Robot



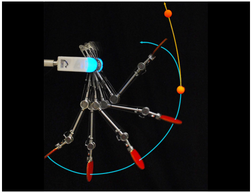
- Autonomously learn block-stacking with a low-cost robot
- Kinect camera as only sensor
- Robot very noisy
- Learn forward model and controller from scratch
- Small number of interactions: **Robot wears out quickly**

Deisenroth et al. (RSS, 2011): *Learning to Control a Low-Cost Manipulator using Data-efficient Reinforcement Learning*

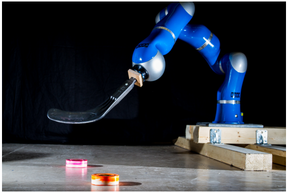
Wide Applicability



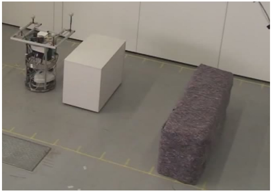
with D Fox



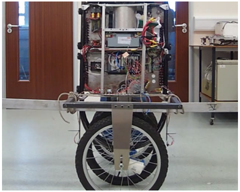
with P Englert, A Paraschos, J Peters



with A Kupcsik, J Peters, G Neumann



B Bischoff (Bosch), ESANN 2013



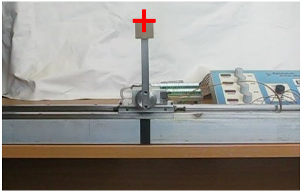
A McHutchon (U Cambridge)

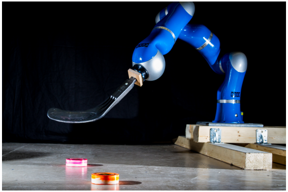
B Bischoff (Bosch), ECML 2013

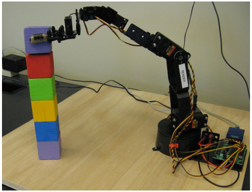
- Application to a wide range of robotic systems

Deisenroth et al. (RSS, 2011): *Learning to Control a Low-Cost Manipulator using Data-efficient Reinforcement Learning*
Englert et al. (ICRA, 2013): *Model-based Imitation Learning by Probabilistic Trajectory Matching*
Deisenroth et al. (ICRA, 2014): *Multi-Task Policy Search for Robotics*
Kupcsik et al. (AIJ, 2017): *Model-based Contextual Policy Search for Data-Efficient Generalization of Robot Skills*

Summary (1)

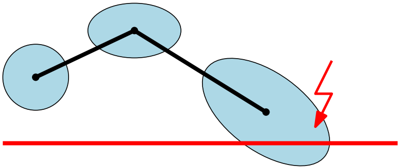




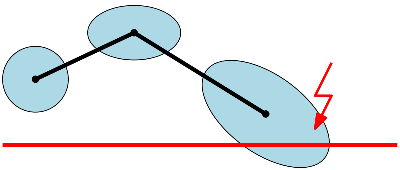


- In robotics, data-efficient learning is critical
- Probabilistic, model-based RL approach
  - Reduce model bias
  - Unprecedented learning speed
  - Wide applicability

Safe Exploration



- Deal with real-world safety constraints (states/controls)
- Use probabilistic model to predict whether state constraints are violated (e.g., Sui et al., 2015; Berkenkamp et al., 2017)
- Adjust policy if necessary (during policy learning)



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▸▸ Safe exploration within an MPC-based RL setting
▸▸ Optimize control signals $\boldsymbol{u}_t$ directly (no policy parameters)

Approach

- Idea: Optimize control signals directly (instead of policy parameters)
- Few parameters to optimize ⏩ Low-dimensional search space
- Open-loop control
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Approach

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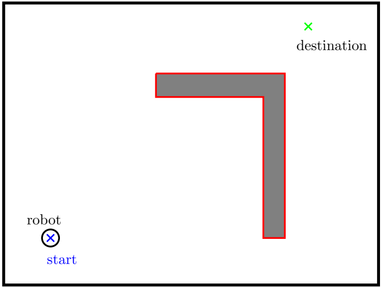
- Few parameters to optimize ⏩ Low-dimensional search space
- Open-loop control
  ⏩ No chance of success (with minor model inaccuracies)
- Model Predictive Control (MPC) turns this into a closed-loop control approach
- Positive side-effect: Increase robustness to model errors (online approach) ⏩ Increase data efficiency

Model Predictive Control



- Given a state x_t, plan (open loop) over a short horizon of length H to get an open-loop control sequence $\boldsymbol{u}^*_{t+0}, \dots, \boldsymbol{u}^*_{t+H-1}$
- After transitioning into a new state x_{t+1}, re-plan (as previously): Get $\boldsymbol{u}^*_{t+1+0}, \dots, \boldsymbol{u}^*_{t+1+H-1}$ ▶▶ closed-loop/feedback control

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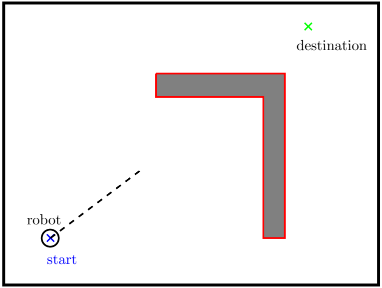

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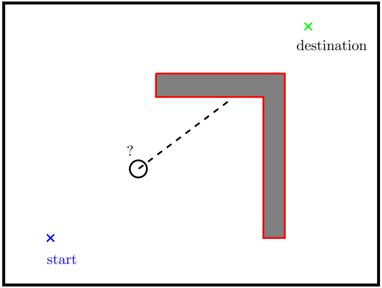

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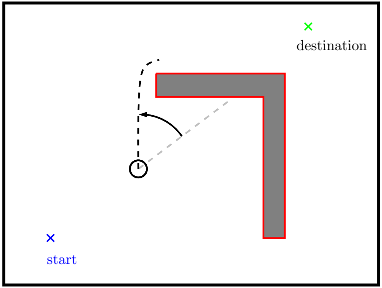

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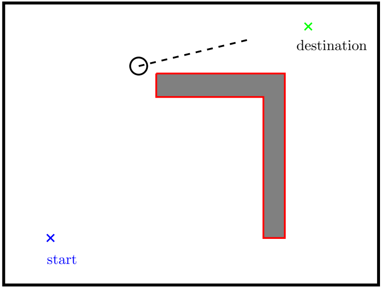

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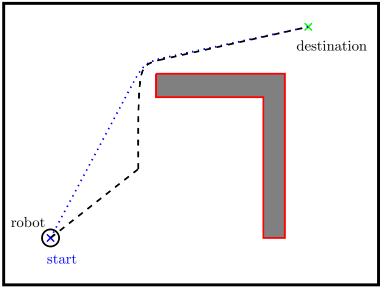

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- Use this within a trial-and-error RL setting

Probabilistic MPC in RL

- Learned GP model for transition dynamics
- Repeat (while executing the policy):
  1. In current state $\boldsymbol{x}_t$, determine optimal control sequence $\boldsymbol{u}_0^*, \ldots, \boldsymbol{u}_{H-1}^*$
  2. Apply first control $\boldsymbol{u}_0^*$ in state $\boldsymbol{x}_t$
  3. Transition to next state $\boldsymbol{x}_{t+1}$
  4. Update GP transition model

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Theoretical Results

- Uncertainty propagation is deterministic (GP moment matching)
  - Re-formulate system dynamics:

$$\boldsymbol{z}_{t+1} = f_{MM}(\boldsymbol{z}_t, \boldsymbol{u}_t)$$

$$\boldsymbol{z}_t = \{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$$ Collects moments

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- Lipschitz continuity (under mild assumptions) implies that we can apply Pontryagin's Minimum Principle
  - Control Hamiltonian $H(\boldsymbol{\lambda}_{t+1}, \boldsymbol{z}_t, \boldsymbol{u}_t)$
  - Adjoint recursion for $\boldsymbol{\lambda}_t$
  - Necessary optimality condition: $\partial H / \partial \boldsymbol{u}_t = \boldsymbol{0}$
  - ⏩ Principled treatment of constraints on controls

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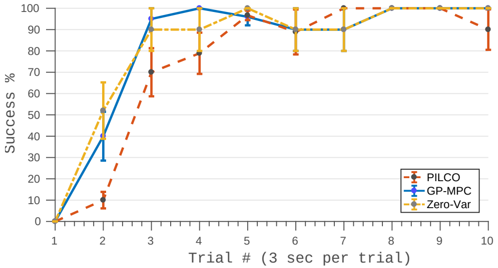
  ⏩ Principled treatment of constraints on controls
- Use predictive uncertainty to check violation of state constraints

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Learning Speed (Cart Pole)

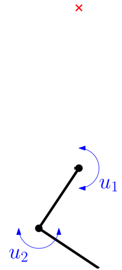


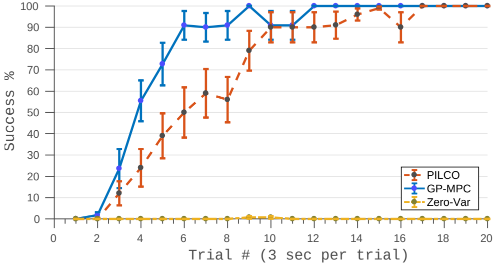
- Zero-Var: Only use the mean of the GP, discard variances for long-term predictions
- MPC: Increased data efficiency (40% less experience required than PILCO)

  ▶▶ MPC more robust to model inaccuracies than a parametrized feedback controller

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Learning Speed (Double Pendulum)





- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
  - Gets stuck in local optimum near start state
  - Insufficient exploration due to lack of uncertainty propagation

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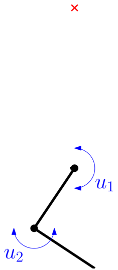
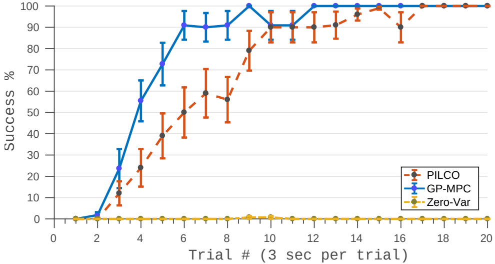
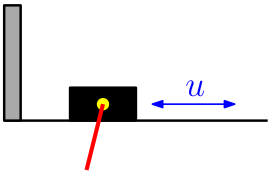


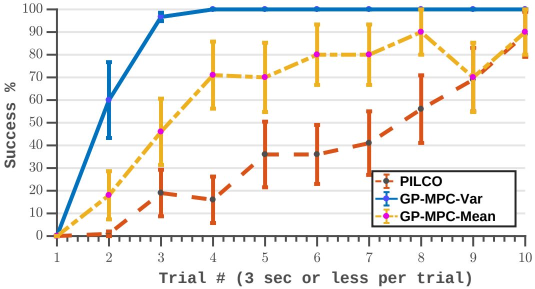


- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
  - Gets stuck in local optimum near start state
  - Insufficient exploration due to lack of uncertainty propagation
- Although MPC is fairly robust to model inaccuracies we cannot get away without uncertainty propagation

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Safety Constraints (Cart Pole)

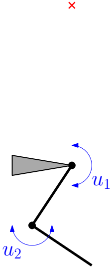


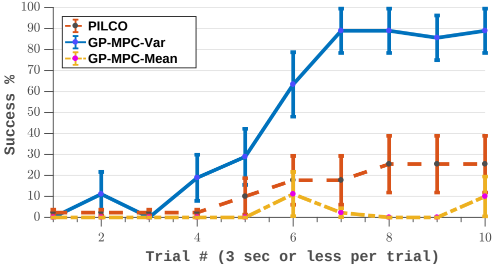


PILCO	16/100	constraint violations
GP-MPC-Mean	21/100	constraint violations
GP-MPC-Var	3/100	constraint violations

Propagating model uncertainty important for safety

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

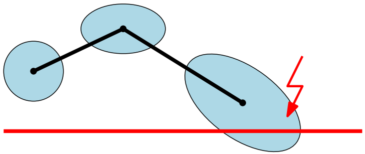


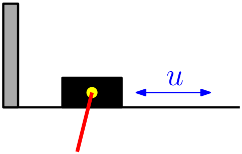


Experiment	Double Pendulum
PILCO	23/100
GP-MPC-Mean	26/100
GP-MPC-Var	11/100

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

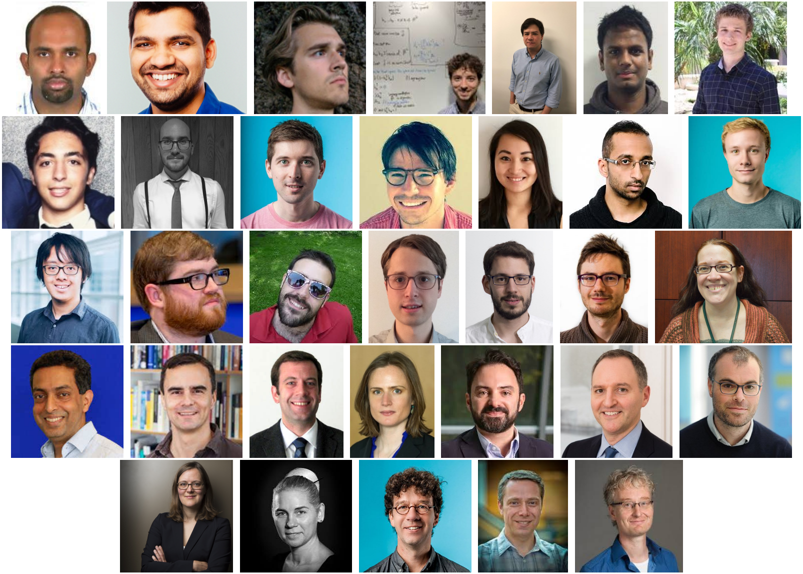
Summary (2)



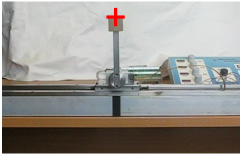
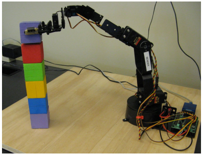


- Probabilistic prediction models for safe exploration
- Uncertainty propagation can be used to reduce violation of safety constraints
- MPC framework increases robustness to model errors
  ▸▸ Increased data efficiency

Team and Collaborators

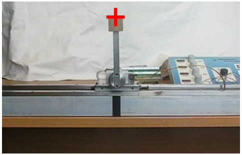


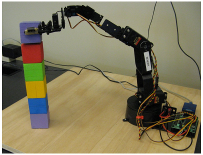
Wrap-up

- In robotics, data-efficient learning is critical
- Controller learning based on learned probabilistic models
  - Reinforcement learning
  - Safe exploration and MPC
- Key to success: Probabilistic modeling and Bayesian inference

Wrap-up





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ありがとうございました

References I

[1] F. Berkenkamp, M. Turchetta, A. P. Schoellig, and A. Krause. Safe Model-based Reinforcement Learning with Stability Guarantees. In *Advances in Neural Information Processing Systems*, 2017.

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GP Moment Matching: Some Details

$$f \sim GP(0, k)\,, \quad \text{Training data: } \boldsymbol{X}, \boldsymbol{y}$$
$$\boldsymbol{x}_* \sim \mathcal{N}\big(\boldsymbol{\mu},\, \boldsymbol{\Sigma}\big)$$

- Compute $\mathbb{E}[f(\boldsymbol{x}_*)]$

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- Compute $\mathbb{E}[f(\boldsymbol{x}_*)]$

$$\mathbb{E}_{f,\boldsymbol{x}_*}[f(\boldsymbol{x}_*)] = \mathbb{E}_{\boldsymbol{x}}\big[\,\mathbb{E}_f[f(\boldsymbol{x}_*)|\boldsymbol{x}_*]\,\big] = \mathbb{E}_{\boldsymbol{x}_*}[\,m_f(\boldsymbol{x}_*)\,]$$

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&= \mathbb{E}_{\boldsymbol{x}_*}\big[\,k(\boldsymbol{x}_*, \boldsymbol{X})(\boldsymbol{K} + \sigma_n^2\boldsymbol{I})^{-1}\boldsymbol{y}\,\big] \\
&= \boldsymbol{\beta}^\top \int k(\boldsymbol{X}, \boldsymbol{x}_*)\mathcal{N}\big(\boldsymbol{x}_* \,|\, \boldsymbol{\mu},\, \boldsymbol{\Sigma}\big) d\boldsymbol{x}_*
\end{aligned}$$

$\boldsymbol{\beta} := (\boldsymbol{K} + \sigma_n^2\boldsymbol{I})^{-1}\boldsymbol{y}$ ⏩ independent of $\boldsymbol{x}_*$

GP Moment Matching: Some Details

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\end{aligned}$$

$$\boldsymbol{\beta} := (\boldsymbol{K} + \sigma_n^2 \boldsymbol{I})^{-1}\boldsymbol{y}$$ ▶▶ independent of $\boldsymbol{x}_*$

- If k is a Gaussian (squared exponential) kernel, this integral can be solved analytically
- Variance of $f(\boldsymbol{x}_*)$ can be computed similarly